4. Voltage Induction in Three-Phase Machines

4.1 *FARADAY´*s Law (1831)

a) Stationary Induction:
The **second law of *MAXWELL***

$$curl\vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.1}$$

describes the voltage induction in windings. Applying *STOKES´*s law of integrals

$$\oint_C \vec{E}\cdot d\vec{s} = \int_A curl\vec{E}\cdot d\vec{A}$$

for a conductor loop (closed loop *C* in Fig. 4.1a) that encloses the area *A*, equation (4.1) is converted from differential into integral form.

$$u_i = \oint_C \vec{E}\cdot d\vec{s} = \int_A rot\vec{E}\cdot d\vec{A} = -\int_A \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A} \tag{4.2}$$

The electrically induced voltage along the loop *C* (integral of the electric field strength) is called u_i. The partial derivation $\partial/\partial t$ can be put in front of the integral. This is possible with a constant area *A*, hence no dependence *A* of *t*, which is only possible with a **stationary conductor loop is considered**:

$$\boxed{u_i = \oint_C \vec{E}_{Wi}\cdot d\vec{s} = -\frac{\partial}{\partial t}\int_{A=const.} \vec{B}\cdot d\vec{A} = -\frac{\partial \Phi}{\partial t}}. \tag{4.3}$$

In equation (4.3), Φ is the magnetic flux penetrating the area *A*. According to the **law of induction** for **<u>stationary</u>** current loops (4.3), it is:
The electric curl field $\vec{E}_{Wi}$ and the associated induced voltage u_i is induced by the **negative** variation with time of the magnetic flux Φ enclosed by the "conductor loop" *C*. Therefore, the direction of $\vec{E}_{Wi}$ is to the **left** of the direction of $\partial\vec{B}/\partial t$ (left-hand-rule). As the current loop is stationary, this phenomenon is called **"stationary induction"**.

It is important to distinguish carefully between the electric **curl field** (eddy field) E_{Wi} and the **source field** E_{Qu}. According to (4.1), the electrically induced field is a curl field $\vec{E}_{Wi}$, because it is calculated by a "curl" operation. The field lines are closed loops (subscript "wi", Fig. 4.1a). This field causes a displacement of electric charges along the conductor loop *C*, which is open at the terminals 1 and 2, in a way that terminal 1 becomes negatively, and terminal 2 becomes positively charged for a positive $\partial\Phi/\partial t$. These charges (charge density ρ) generate an electric source field $\vec{E}_{Qu}$ according to *MAXWELL*'s 4[th] law $div\vec{E}_{Qu} = \rho/\varepsilon_0$. $\vec{E}_{Qu}$ can be measured between the terminals 1 and 2.

b) Motion Induction:
If the current loop *C* **moves** with the velocity *v* in the magnetic field, the flux passing the area *A* may also change, even in the case of *B* being CONSTANT with time (*B* = const.). Two cases can be distinguished:
1. The **shape** of the loop changes, and so does the area *A* (Fig. 4.1b).

2. The loop moves from a region with flux density B_1 to a region with a **spatial different** flux density $B_2 \neq B_1$.

In both cases, the moving of the loop within a magnetic field which is constant with time causes a **changing of the flux across the area enclosed by the loop *C***. According to (4.3), a voltage u_i is induced.

In both cases 1. and 2., the area *A* is not constant, because the coordinates (*x,y,z*) of the loop *C*(*x,y,z*) change with time: *x*(*t*), *y*(*t*), *z*(*t*). If the derivation with respect to time is put IN FRONT of the integral (as in (4.3)), the change of the area *A* has to be taken into account (product rule!). As

$$\partial/\partial t(x,y,z) = (\partial x/\partial t, \partial y/\partial t, \partial z/\partial t) = (v_x, v_y, v_z) = \vec{v}$$

is the **vector of the velocity** of each path element $d\vec{s}$ of the conductor loop *C*, we get according to vector analysis

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_A \vec{B}\cdot d\vec{A} = \frac{\partial}{\partial t}\int_{A=cst.} \vec{B}\cdot d\vec{A} - \oint_C (\vec{v}\times\vec{B})\cdot d\vec{s} = \int_{A=cst.} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A} - \oint_C (\vec{v}\times\vec{B})\cdot d\vec{s} \quad . \tag{4.4}$$

As *B* is constant, it is $\partial\vec{B}/\partial t = 0$. Combining (4.4) and (4.3), we get

$$\boxed{u_i = \oint_C \vec{E}_b\cdot d\vec{s} = \oint_C (\vec{v}\times\vec{B})\cdot d\vec{s}}. \tag{4.5}$$

The **law of induction** for <u>moving</u> current loops (4.5) signifies:
Moving of a conductor loop *C* with the velocity $\vec{v}$ in the field *B*, which is constant with time, generates an electrical **field density induced by the motion** $\vec{E}_b = \vec{v}\times\vec{B}$. The integral along the loop *C* gives the motion induced voltage u_i. As the conduction loop moves, this phenomenon is called **"motion induction"**.

c) General Law of Induction:
- If a current loop *C* moves with the velocity *v* in the magnetic field *B*, where
- *B* changes with time,

stationary and motional induction occur at the same time.

$$\boxed{u_i = \oint_C \left(\vec{E}_{Wi} + \vec{E}_b\right)\cdot d\vec{s} = \oint_C \left(\vec{E}_{Wi} + \vec{v}\times\vec{B}\right)\cdot d\vec{s} = -\frac{d\Phi}{dt}} \tag{4.6}$$

The **general law of induction** (4.6) signifies:
Every variation of the flux Φ through the area *A* that is enclosed by the conductor loop *C* induces a voltage u_i; this induced voltage equals the negative rate of change with time of the linked flux.

If the loop has *N* turns in series, u_i is *N*-times larger: $u_i = -N\cdot d\Phi/dt$. With the **flux linkage**

$$\boxed{\Psi = N\cdot\Phi = N\cdot\int_A \vec{B}\cdot d\vec{A}} \tag{4.7}$$

the **most general expression of the law of induction** is obtained:

$$\boxed{u_i = -\frac{d\Psi}{dt}} \tag{4.8}$$

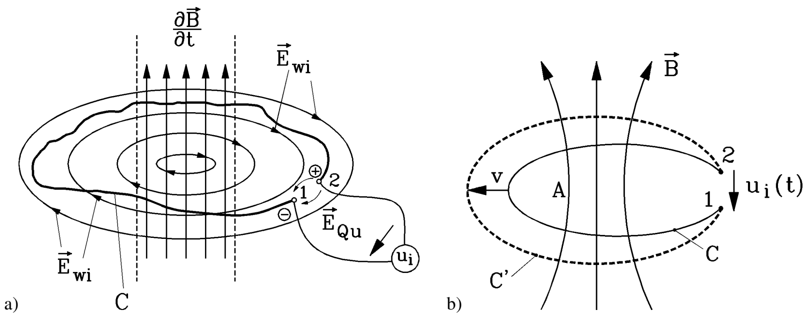


Fig. 4.1: **Voltage induction** in a current loop *C:* a) stationary current loop, but variation of the magnetic field with time (stationary induction). b) the current loop moves within a magnetic field that is constant with time (motion induction)

d) Current Loop at Zero Current:
At open terminals 1 and 2, no current may flow within the current loop *C*. Hence, the resultant force on the charge carriers (with charge *Q*) is zero.

$$\vec{F} = Q \cdot \vec{E} = Q \cdot (\vec{E}_{Wi+b} + \vec{E}_{Qu}) = 0 \quad \Rightarrow \quad \vec{E}_{Wi+b} = -\vec{E}_{Qu} \tag{4.9}$$

As the terminals 1 and 2 are immediate neighbours, the integral along the path *C* from 1 to 2 equals the integral of the closed loop: $\int_1^2 \ldots \cdot d\vec{s} = \oint_C \ldots \cdot d\vec{s}$. Therefore, the induced voltage is

$$u_i = \oint_C \left(\vec{E}_{Wi} + \vec{v}x\vec{B}\right) \cdot d\vec{s} \approx \int_1^2 \left(\vec{E}_{Wi} + \vec{v}x\vec{B}\right) \cdot d\vec{s} = \int_1^2 \vec{E}_{Wi+b} \cdot d\vec{s} \quad . \tag{4.10}$$

Accordingly, the voltage that is measurable from terminal 2 to terminal 1

$$u_{21} = \int_2^1 \vec{E}_{Qu} \cdot d\vec{s} = -\int_1^2 \vec{E}_{Qu} \cdot d\vec{s} = \int_1^2 \vec{E}_{Wi+b} \cdot d\vec{s} = u_i \tag{4.11}$$

equals the induced voltage.

e) Current Loop at Load:
If an external, ideal voltage source *u* (without any internal resistance) is connected, a current *i* flows. This current is both driven by the induced voltage and by the external voltage source and is only limited by the internal resistance *R* of the loop.

$$R \cdot i = u + u_i = u - \frac{d\Psi}{dt} \qquad \Rightarrow \qquad R \cdot i + \frac{d\Psi}{dt} = u \tag{4.11}$$

If the current *i* that excites the magnetic field *B* **flows in the loop itself**, the factor between the flux linkage and the magnetic field is called **self inductance** *L*.

$$\boxed{\Psi(t) = L \cdot i(t)} \quad , \tag{4.12}$$

If a second current i_2 **flows in a second loop** and excites the field, which is linked to the considered (first) loop, the factor is called **mutual inductance** *M*

$$\boxed{\Psi(t) = M \cdot i_2(t)} \quad . \tag{4.13}$$

As a result, (4.11) can be rewritten to give the common expression for an inductive circuit:

$$R \cdot i + L\frac{di}{dt} = u \qquad\qquad R \cdot i_1 + M\frac{di_2}{dt} = u \tag{4.14}$$

Example 4.1-1:
Shorted stationary coil in a time variant external field *B*:
The terminals 1 and 2 of loop *C* of Fig. 4.1a are connected, hence, they are **shorted** (Fig. 4.2a). The time variant external field *B* causes an increase of the field from the bottom up through the loop area *A* and **induces** an eddy field E_{Wi}. The eddy field E_{Wi} which is connected in **left hand sense** with the direction of change of *B* induces a **short-circuit current** *i* flowing in left hand sense. According to ***AMPERE*'s law**, this current excites a field B_e that is linked to the current *i* in **right hand sense** (right-hand-rule). Hence B_e acts **against** the cause of the induced voltage which is the variation of the magnetic flux density with time $\partial B / \partial t$. So, the field B_e "tries" **to retard** the changing of the external field *B*.**"*LENZ*' law": The current that flows as a result of the induced voltage u_i generates a self-field B_e, which acts against the cause of u_i (the change of the field *B*).**

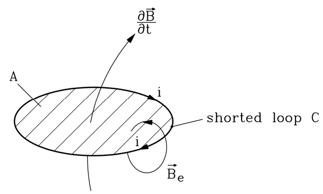


Fig. 4.2a: The voltage u_i that is induced in a shorted loop due to variation of the magnetic flux density causes a current *i*. The self-field of *i* B_e acts against the variation of the field.

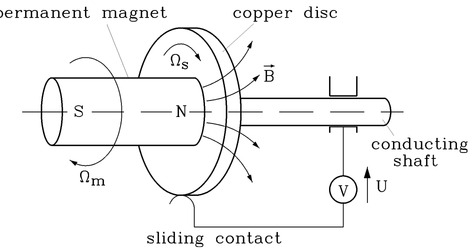


Fig. 4.2b: *FARADAY´*s disc: rotating copper disc in a magnetic field

Example 4.1-2: ***FARADAY´*s disc** (Fig. 4.2b):
*FARADAY´*s disc is a simple model for a unipolar machine which generates a perfectly constant voltage (dc voltage) without use of any electric or mechanical rectifier. It is a conducting disc (e.g. copper) that is rotating together with a conducting shaft (bearing not shown). The induced voltage due to the disc motion in the constant field *B* is measured between two sliding contacts, one at the shaft and one at the outer edge of the wheel. Further, the bar magnet is arranged co-axially with the disc. It may rotate independently of the disc.

Case a)
If the disc rotates with angular speed Ω_s and the magnet is at rest ($\Omega_m = 0$), **motion induction** occurs, because the conductor (the disc) moves within the field. The magnet itself does not change its field: *B* = const. *NO* stationary induction does occur: $\partial B / \partial t = 0$. The velocity

$$v = r \cdot \Omega_s \tag{4.15}$$

of each point of the disc at distance *r* of the axis of rotation is perpendicular to the vector of the radius (circumferential direction). The vector of the magnetic flux density $\vec{B}$ leaves the disc in axial perpendicular direction. Accordingly, it is perpendicular to the vector of the velocity $\vec{v}$. Hence, the electric field due to the motion

$$\vec{E}_b = \vec{v}x\vec{B} = v \cdot B \cdot \vec{e}_r \quad (4.16)$$

is oriented in radial direction pointing outwards ($\vec{e}_r$: unity vector in radial direction). The outer edge of the disc becomes positively charged. A source field E_{Qu} that has the same magnitude but is oriented in the opposite direction (inwardly) is generated. The path integral of E_{Qu} can be measured as dc-voltage at the sliding contacts.

$$U = \int_{r_a}^{r_i} \vec{E}_{Qu} \cdot d\vec{s} = -\int_{r_a}^{r_i} \vec{E}_b \cdot d\vec{s} = \int_{r_i}^{r_a} \vec{E}_b \cdot d\vec{s} = \int_{r_i}^{r_a} v \cdot B \cdot dr \quad (4.17)$$

If B is constant along the radius $B(r) = B$, (4.18) can be derived from (4.17):

$$U = \int_{r_i}^{r_a} v \cdot B \cdot dr = \int_{r_i}^{r_a} r \cdot \Omega_s \cdot B \cdot dr = \Omega_s \cdot B \cdot \left(r_a^2 - r_i^2\right)/2 \quad (4.18)$$

Result:
The induced voltage increases proportionally with the frequency of rotation of the disc and with the magnetic flux density.

Case b)
If the disc is at rest ($\Omega_s = 0$) and the magnet rotates ($\Omega_m > 0$), NO voltage proportional to $B\Omega_m$ is induced. NO change of the magnetic flux enclosed by the disc occurs, because the magnetic field is constant with time and, because of the rotational symmetry of the bar magnet, it is also constant along the circumference angle γ. Therefore, as long as the disc is at rest, **no voltage is induced**, independently from the rotational speed of the magnet ($\Omega_s = 0$ and $\Omega_m > 0$). The reason for a misapprehension of the occurrence of an induced voltage proportional to $B\Omega_m$ is the wrong interpretation of the flux lines (flux tubes) as material entities with an observable relative motion, instead of the right interpretation as a mathematical model, which indicate the magnitude and the direction of the field.

f) Relevance of the Law of Induction:
The law of induction is of utmost importance for the functioning of electric machines and transformers.

Stationary induction	*Motion induction*
field of B variable with time	field B constant with time
coil at rest	coil moves with velocity v
$u_i = -d\Psi / dt$	
$u_i = -\partial\Psi / \partial t = \oint \vec{E}_{Wi} \cdot d\vec{s}$	$u_i = \oint \left(\vec{v} \times \vec{B}\right) \cdot d\vec{s} = \oint \vec{E}_b \cdot d\vec{s}$
application of the law of induction:	
• transformer coils • stator windings of three-phase machines	• rotating armature winding of dc-machines
transformer induction	*rotary induction*

Table 4.1: Relevance of the law of induction in electric machines and transformers

Example 4.1-3:
The general law of induction can ALWAYS be applied !

This is shown with a simple linear machine (Fig. 4.3):
One coil (number of turns N_c, coil width τ) moves with the velocity v in the air gap between an iron yoke and permanent magnets (pole sequence N-S-N-S, pole width $b_p = \tau$). The air gap field B_δ is homogeneously positive or negative, depending on the polarity of the magnet.

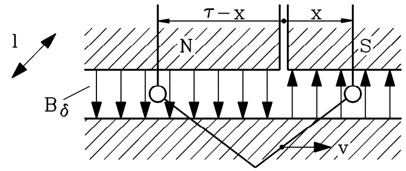


Fig. 4.3: Voltage induction in a coil (coil width τ, velocity v) that moves in a magnetic field B_δ that is constant with time.

a) Calculation of u_i using the law of motion induction:
The induction at stand-still is zero, because the field of the permanent magnets does not change. Voltage can only be induced by motion. The loop C becomes the length $2l$, because the face ends of the coil are outside of the area with a magnetic field. As the directions of velocity, field and orientation of the coil sides are perpendicular to each other, it is:

$$u_i = N_c \cdot 2\int_0^l (\vec{v}x\vec{B}) \cdot d\vec{s} = \underline{2N_c vBl} \quad . \quad (4.19)$$

b) Calculation of u_i using the general law of induction:
The flux linkage Ψ changes due to the motion of the coil within the magnetic field, because the co-ordinate of the location $x = vt$ changes with time.
Flux linked with the coil: $\Phi = \int_A \vec{B} \cdot d\vec{A} = l \cdot \left[(\tau - x)B_\delta - xB_\delta\right] = lB_\delta(\tau - 2x)$
Flux linkage of the coil: $\Psi(t) = N_c\Phi(t) = N_c lB_\delta(\tau - 2x) = N_c lB_\delta(\tau - 2 \cdot v \cdot t)$

Application of the general law of induction:

$$u_i = -\frac{d\Psi}{dt} = -N_c lB_\delta \cdot \frac{d(\tau - 2 \cdot v \cdot t)}{dt} = \underline{2N_c vB_\delta l} \quad (4.20)$$

Results:
Equation (4.20) and (4.19) are equal. Explanation: An observer that moves with the coil would not be able to assert the movement of the coil. He would only realise a change of the flux linkage without knowing the reason for it.

The decomposition in stationary and motion induction depends on the position of the observer. ***Ultimately, only the total variation of the flux linkage is important for the calculation of the induced voltage.***

4.2 Voltage Induction in a Stator Coil

a) Voltage Induction due to the Fundamental of the Field:
Fig. 4.4 shows a stationary, full-pitched coil that is inserted into the stator slots (length per slot: l). We assume, the field outside the area of the air gap as zero.

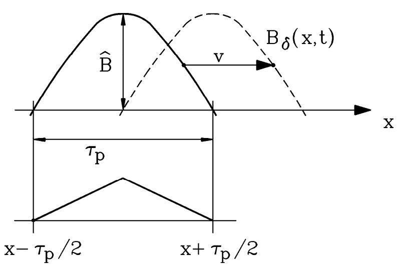


Fig. 4.4: A sinusoidal travelling wave induces a stationary coil, $W = \tau_p$

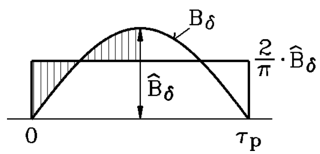


Fig. 4.5: Flux of a pole with sinusoidal distributed flux density, interpreted as area beneath the field distribution

Accordingly, no additional flux is enclosed by the end connections of the coil. The fundamental of the radial component of the flux density as described in Chapter 3

$$\boxed{B_{\delta 1}(x,t) = \hat{B}_{\delta 1}\cos\left(\frac{x\pi}{\tau_p} - \omega t\right)} \qquad (4.21)$$

generates an alternating flux in the coil

$$\Phi(t) = l\int_{-\tau_p/2}^{\tau_p/2} B_{\delta 1}(x,t)dx = \frac{2}{\pi}\tau_p l\hat{B}_{\delta 1}\cdot\cos\omega t\ . \qquad (4.22)$$

The amplitude of this flux is (Fig. 4.5)

$$\boxed{\Phi_c = \frac{2}{\pi}\tau_p l\hat{B}_{\delta 1}}\ . \qquad (4.23)$$

It pulsates with the frequency $f = \omega/(2\pi)$ (Fig. 4.5). The flux linkage pulsates with the same frequency

$$\Psi_c(t) = N_c\Phi_c\cdot\cos\omega t\ , \qquad (4.24)$$

thereby **inducing a sinusoidal, alternating voltage in the coil**:

$$u_{i,c}(t) = -\frac{d\Psi_c(t)}{dt} = \hat{U}_{i,c}\sin\omega t \qquad (4.25)$$

The amplitude of this voltage is

$$\boxed{\hat{U}_{i,c} = \omega N_c\Phi_c = 2\pi f N_c\frac{2}{\pi}\tau_p l\hat{B}_{\delta 1}}\ . \qquad (4.26)$$

b) Voltage Induction due to Rotating Fields of Harmonics:
The field that is induced by the excited rotor (rotating with speed *n*) of a synchronous generator is expressed as *FOURIER*-sum of the individual sinusoidal field waves (see Chapter 3).

Generally, the inducing field waves are expressed with the ordinal numbers: the **fundamental**: $\mu = 1$, and of the **harmonics**: $\mu > 1$.

$$B_{\delta,\mu}(x,t) = \hat{B}_{\delta\mu}\cos\left(\frac{\mu x\pi}{\tau_p} - \mu\cdot\omega\cdot t\right)\ ,\ \mu = 1, 3, 5, 7, \ldots\ . \qquad (4.27)$$

where $\omega = 2\pi\cdot n\cdot p$. The alternating flux that is inducing a stationary coil in the stator is

$$\Phi_{c\mu}(t) = l\int_{-\tau_p/2}^{\tau_p/2} B_{\delta,\mu}(x,t)dx = \frac{2}{\pi}\cdot\frac{\tau_p}{\mu}l\hat{B}_{\delta\mu}\cdot\sin\left(\frac{\mu\pi}{2}\right)\cdot\cos(\mu\omega t) \qquad (4.28)$$

with the amplitude

$$\Phi_{c\mu} = \frac{2}{\pi}\cdot\frac{\tau_p}{\mu}l\hat{B}_{\delta\mu}\ . \qquad (4.29)$$

The amplitude of the flux of the harmonics is by $\hat{B}_{\delta\mu}/(\mu\cdot\hat{B}_{\delta 1})$ **smaller** than the flux of the fundamental, but it pulsates with a **significantly higher frequency** $f_\mu = \mu\omega/(2\pi)$. The expression $\sin(\mu\pi/2) = (-1)^{(\mu-1)/2}$ (with $\mu = 1, 3, 5, \ldots$) values always only 1, -1, 1, -1, ... Only the sign changes, but not the amplitude. The voltage induced in a single stationary coil in the stator is,

$$u_{i,c,\mu} = -\frac{d\Psi_{c\mu}}{dt} = -N_c\frac{d\Phi_{c\mu}}{dt} = \mu\omega\cdot N_c\cdot\frac{2}{\pi}\cdot\frac{\tau_p}{\mu}l\hat{B}_{\delta\mu}\cdot\sin\left(\frac{\mu\pi}{2}\right)\cdot\sin(\mu\omega t) \qquad (4.30)$$

The particular case $\mu = 1$ equals (4.25), (4.26).

Result:
Not only the "useful" fundamental voltage with the frequency $f = n\cdot p$, but also additional, harmonic voltages are induced in the stator coils of a "real" machine. The amplitudes of these harmonics voltages are smaller than the amplitude of the "useful" fundamental, yet, the frequencies are higher.

Example 4.2-1:
12-pole synchronous generator: $n = 500$/min, $2p = 12$, $f = n\cdot p = (500/60)\cdot 6 = 50\,\text{Hz}$
stator coil data: $N_c = 2$, $W = \tau_p = 0.5$ m, $l = 1$ m
Induced voltage at amplitudes of the rotor field that is inducing the coil according to Table 4.2:

μ	$\hat{B}_{\delta\mu}$	f_μ	$\Phi_{c\mu}$	$U_{i,c\mu} = \hat{U}_{i,c\mu}/\sqrt{2}$	$U_{i,c\mu}/U_{i,c1}$
-	T	Hz	mWb	V	%
1	0.9	50	286.5	127.2	100
3	0.15	150	-15.9	-21.2	16.7
5	0.05	250	3.3	7.1	5.6
7	0.05	350	-2.3	-7.1	5.6

Table 4.2: Example of voltage induction in a stator winding due to fundamental and harmonic rotor fields

4.3 Voltage Induction in a Three-Phase Winding

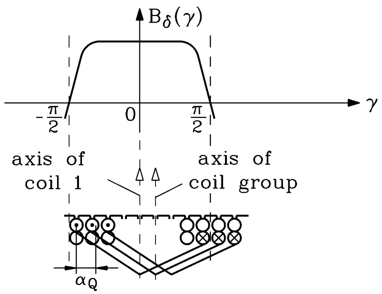


Fig. 4.6: Induction of a short-pitched coil of arbitrary span W due to a magnetic field. The figure shows a coil group with $q = 3$ coils.

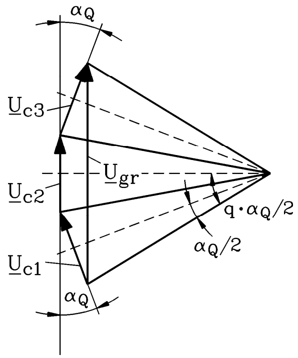


Fig. 4.7: Three fully-pitched, series-connected coils form a coil group. The voltage per group $\underline{U}_{gr}$ is determined by means of complex phasors of alternating voltages.

A poly-phase winding consists of m phases, where each phase is a series connection (or parallel connection) of winding branches built from individual coils. Generally, the coils are short-pitched in case of a two-layer winding, hence, the coil span W is smaller than the pole pitch.

a) Voltage Induction in Short-Pitched Coils:

In Fig. 4.6, the short-pitching is indicated by $\beta = (W/\tau_p)\pi$. The flux linkage of a **short-pitched** coil is by the **pitch factor** $k_{p,\mu}$ smaller than the flux linkage of a fully-pitched coil, as equation (4.31) shows when compared with equation (4.28).

$$\Phi_{c\mu}(t) = l \int_{-W/2}^{W/2} \hat{B}_{\delta\mu} \cos(\frac{\mu\pi x}{\tau_p} - \mu\omega t)dx = \frac{2}{\pi\mu}\tau_p l \hat{B}_{\delta\mu} \cdot \sin(\mu\frac{\pi}{2}\frac{W}{\tau_p}) \cdot \cos\mu\omega t \qquad (4.31)$$

$$\boxed{k_{p,\mu} = \sin\left(\mu\frac{\pi}{2}\cdot\frac{W}{\tau_p}\right)} \qquad (4.32)$$

b) Voltage Induction in a Coil Group:

How big is the induced voltage of a coil group consisting of q fully-pitched coils that are – series-connected – arranged in q neighbouring slots (slot pitch τ_Q)? The fundamental and the harmonic waves induce sinusoidal alternating voltages in each coil according to (4.30). The individual coil voltages of the fundamental are phase shifted by the slot angle (Fig. 4.6 and Fig. 4.7)

$$\alpha_Q = \frac{2\pi}{2mq} \qquad (4.33)$$

At the time of maximum flux linkage of the first coil of the coil group, when the maximum of the field of the sinusoidal fundamental is in the centre of the coil, this maximum is at one slot pitch τ_Q distance from the centre of the second coil. After the time $t = \tau_Q/v$ the maximum of

the field wave has moved to in the centre of the second coil. With $v = 2f\tau_p$ and $\tau_p/\tau_Q = mq$ (e.g. $q = 3$, $m = 3$: $\tau_p/\tau_Q = 9$ slots per pole), the time t becomes $t = 1/(2fmq)$. This equals a phase shift of

$$\alpha_Q = \omega \cdot t = \omega/(2fmq) = 2\pi f/(2fmq) = 2\pi/(2mq). \qquad (4.34)$$

In the case of voltage induction by a μ^{th} harmonic, the phase difference is by the factor μ larger, as the frequency of induction is μ times as large.

$$\alpha_{Q,\mu} = \mu \cdot \omega \cdot t = \mu \cdot 2\pi/(2mq) \qquad (4.35)$$

This can be understood as follows: The wave length of the μ^{th} harmonic is by the factor $1/\mu$ smaller than the wave length of the fundamental. Therefore, the distance between two neighbouring coils – the slot pitch τ_Q – is μ times as large in terms of wave lengths.

Result:

The induced voltage of a coil group equals the sum of q coil voltages that are shifted by the phase angle $\alpha_{Q,\mu}$ (Fig. 4.7).

Fig. 4.7 shows the induced coil voltages and their sum for $q = 3$. The ratio of the length of the phasor of the geometric sum $\hat{U}_{i,gr,\mu}$ and the algebraic sum of the phasor of a coil voltage $\hat{U}_{i,c,\mu}$ can be derived from the figure:

$$k_{d,\mu} = \frac{\hat{U}_{i,gr,\mu}}{q\hat{U}_{i,c,\mu}} = \frac{2\sin\left(q\frac{\alpha_{Q,\mu}}{2}\right)}{q\cdot 2\sin\left(\frac{\alpha_{Q,\mu}}{2}\right)} = \frac{\sin\left(\mu\frac{\pi}{2m}\right)}{q\cdot\sin\left(\mu\frac{\pi}{2mq}\right)} \qquad (4.36)$$

Equation (4.36) equals the **distribution factor** as it was introduced in chapter 3**.**

Result:

The induced voltage of a coil group is by the distribution factor smaller than the induced voltage of an individual coil with the same number of turns as the coil group.

c) Voltage Induction in the Phase of a Winding:

A machine with $2p$-poles and a **two-layer winding** has $2p$ coil groups, each with q generally short-pitched coils. Due to the short-pitching and the coil groups ($q > 1$), both pitch and distribution factor have to be taken into consideration, giving the **winding factor**. For the fundamental $\mu = 1$, it is:

$$\boxed{k_{w1} = k_{d1} \cdot k_{p1}} \qquad (4.37)$$

Using the number of turns N per phase, the r.m.s.-value of the stator voltage induced by the **field fundamental of the rotor** per phase is – in analogy to (4.26) – given by (4.38).

$$\boxed{U_{i1} = \sqrt{2}\pi f \cdot N \cdot k_{w1} \cdot \frac{2}{\pi}\tau_p l \hat{B}_{\delta 1}} \qquad (4.38)$$

In analogy to (4.30), the phase voltages induced by the μ^{th} harmonic of the rotor field is with $k_{w,\mu} = k_{p,\mu} k_{d,\mu}$

$$U_{i,\mu} = \sqrt{2}\pi\mu f \cdot N \cdot k_{w,\mu} \cdot \frac{2}{\pi}\frac{\tau_p}{\mu} l\hat{B}_{\delta\mu} \quad . \tag{4.39}$$

Example 4.3-1:
- 12-pole synchronous generator: n = 500/min, $2p$ = 12, f = 50 Hz
- stator winding: N_c = 2, q = 2, $W = 5/6\,\tau_p$, a = 1, τ_p = 0.5 m, l = 1 m
- number of turns per phase: $N = 2pqN_c / a = 12 \cdot 2 \cdot 2/1 = 48$

μ	$\hat{B}_{\delta\mu}$	f_μ	$\Phi_{c\mu}$	$U_{i,\mu}$	$U_{i,\mu}/U_{i,1}$
-	T	Hz	mWb	V	%
1	0.9	50	276.7	2850.1	100
3	0.15	150	-11.3	-254.6	8.9
5	0.05	250	0.8	11.4	0.4
7	0.05	350	-0.6	-11.4	0.4

Table 4.3: Example of voltage induction in a phase of a winding due to fundamental and harmonic fields

The amplitudes of the field of the rotor are given according to Table 4.2. The induced phase voltages of the harmonics are strongly reduced by the pitch and distribution factors.

Result:
When compared with the voltage of a fully-pitched winding (Ex. 4.2-1), the 5th and the 7th harmonic of the voltage is reduced from 5.6% to 0.4% by the short-pitching. The 3rd harmonic is also reduced but is still remarkable (8.9% of the fundamental). By using Y-connected stator windings, also the 3rd harmonic (and multiples) are eliminated in the line-to-line voltage.

Example 4.3-2: *Third harmonic phase voltages:* $U_{3U} = U_3 \cos(3\omega t)$

$$U_{3V} = U_3 \cos(3\omega t + 2\pi/3) = U_3 \cos(3\omega t) = U_{3U}$$
$$U_{3W} = U_3 \cos(3\omega t + 4\pi/3) = U_3 \cos(3\omega t) = U_{3U}$$

Third harmonic line-to-line voltage is zero: $U_{LL} = U_{U-V} = U_{3U} - U_{3V} = 0\,!$

4.4 Self-Induction per Phase in a Three-Phase Winding

a) Self-Induction of the Stator Rotating Field:
A three-phase machine with constant air gap δ, the rotor without any winding, and a poly-phase stator winding with m phases is considered. As shown in Chapters 2 and 3, this winding excites a step-like rotating field in the air gap (assuming $\mu_{Fe} \to \infty$) when fed with a symmetrical three-phase current system with the frequency f and the r.m.s.-value I (current per phase). This can be expressed by a *FOURIER* sum of sinusoidal rotating waves. These waves with the ordinal numbers ν are rotating waves with – depending on the ordinal number – **positive** or **negative sequence**.

$$B_{\delta\nu}(x,t) = \hat{B}_{\delta\nu} \cdot \cos\left(\frac{\nu\pi x}{\tau_p} - \omega t\right) \qquad \hat{B}_{\delta\nu} = \frac{\mu_0}{\delta}\frac{\sqrt{2}}{\pi}\frac{m}{p} N \frac{k_{w,\nu}}{\nu} I \tag{4.40}$$

$\nu = 1 + 2mg\,,\; g = 0, \pm 1, \pm 2, \pm 3, \ldots$

Due to **self-induction**, these rotating stator field waves induce the stator winding. The circumference speed of the waves v_ν is proportional to $1/\nu$ (Chapter 3). Accordingly, the fundamental and the harmonics induce the stator coils uniformly with the frequency *f*.

$$f_\nu = \frac{\omega}{2\pi} = \nu \cdot p \cdot (n/\nu) = p \cdot n = f \tag{4.41}$$

Analogue to (4.39), the r.m.s.–value of the ν^{th} harmonic of the induced phase voltage is:

$$U_{i,\nu} = \sqrt{2}\pi f \cdot N \cdot k_{w,\nu} \cdot \frac{2}{\pi}\frac{\tau_p}{\nu} l\hat{B}_{\delta\nu} \tag{4.42}$$

b) Magnetising Main and Harmonic Leakage Inductance:
Using the amplitude of the harmonic field (4.40), a correlation between the induced voltage and the current per phase is obtained, giving the **reactance** $X_{h\nu} = \omega L_{h\nu}$ of the ν^{th} harmonic.

$$U_{i,\nu} = X_{h\nu} I = \omega L_{h\nu} I \tag{4.43}$$

$$L_{h\nu} = \mu_0 N^2 \frac{k_{w,\nu}^2}{\nu^2}\frac{2m}{\pi^2}\frac{l\tau_p}{p \cdot \delta} \tag{4.44}$$

All self-inductances $L_{h\nu}$ are only caused by the air gap field. This field is also called **"main field"**, because it is responsible for the conversion of electric into mechanical energy and vice versa. Therefore, the self-inductances of the rotating fields ν given by (4.44) are called **magnetising inductances** $L_{h\nu}$ (index h).

As all voltages $U_{i,\nu}$ are induced in the stator winding with the same frequency *f*, they can be summarised as the overall induced voltage $\sum_{\nu=1,-5,7,\ldots}^{\infty} U_{i,\nu}$. As all harmonic waves are excited by the same phase current *I*, an **inductance of the total air gap field** may be defined:

$$L_{h,tot} = \frac{\sum_{\nu=1,-5,7,\ldots}^{\infty} U_{i,\nu}}{\omega I} = \sum_{\nu=1,-5,7,\ldots}^{\infty} L_{h\nu} = (1+\sigma_o) L_{h,\nu=1} \tag{4.45}$$

The important inductance is the **magnetising inductance of the fundamental** L_h

$$L_{h,\nu=1} = \mu_0 N^2 k_w^2 \frac{2m}{\pi^2}\frac{l\tau_p}{p \cdot \delta} = L_h \quad . \tag{4.46}$$

The sum of the magnetising inductances of the harmonics is much smaller, expressed by the factor σ_o (**harmonic leakage factor**). The value of σ_o is usually smaller than 0.05.

$$\sigma_o = \sum_{\nu=1,-5,7,\ldots}^{\infty} \left(\frac{k_{w,\nu}}{\nu \cdot k_{w,1}}\right)^2 - 1 \tag{4.47}$$

Result:
The harmonic air gap fields are not leakage fields, because they may flow through the rotor, causing forces and torques. However, as they are more a parasitic than a useful effect, their self inductance effect is summarized as harmonic leakage inductance $\sigma_O L_h$.

c) Alternative way to derive L_h by considering interaction of the three phases U, V, W:

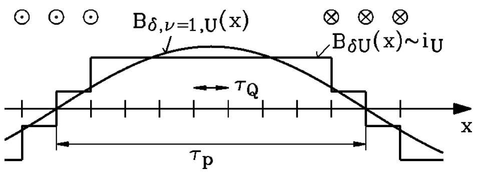


Fig. 4.8: Air gap flux density B_δ and the *FOURIER* fundamental $B_{\delta,\nu=1}$ of the phase winding U with $q = 3$, single-layer winding, one pole is shown

In the strict sense, the self-induction as described under b) is **a combination of self-induction** (e.g. phase U induces in phase U) **and mutual induction** (phase V and W induce in phase U). The field that is excited by the phase U is a **stationary alternating field** that pulsates with the frequency *f*.

According to Chapter 3, the fundamental amplitude of the stationary alternating phase field is

$$B_{\delta,\nu=1} = \frac{\mu_0}{\delta}\frac{\sqrt{2}}{\pi}\frac{2}{p}Nk_{w,\nu=1}I \quad . \tag{4.48}$$

The flux per pole of the fundamental of a fully-pitched coil is

$$\Phi_{c,\nu=1} = \frac{2}{\pi}\tau_p l_{Fe} B_{\delta,\nu=1} , \tag{4.49}$$

and the flux linkage of phase U caused by the phase current i_U:

$$\Psi_{UU} = L_{hUU} \cdot i_U = N \cdot k_{w,\nu=1} \cdot \Phi_{\nu=1} \quad . \tag{4.50}$$

Self-inductance of a phase: $L_{hUU} = \mu_0 N^2 k_{w,1}^2 \frac{4}{\pi^2}\frac{l\tau_p}{p \cdot \delta} = L_{h,ph}$ (4.51)

The flux linkage of the U-coils of the stationary alternating field generated by the current i_V, (hatched area in Fig. 4.9), is – at same current amplitude – only half as large as the flux (4.42), because negative and positive flux components compensate partially.

$$\Psi_{UV} = L_{hUV} \cdot i_V = cos(2\pi/3) \cdot L_{hUU} \cdot i_V = -\frac{L_{hUU}}{2} \cdot i_V \quad . \tag{4.52}$$

In the same way the flux linkage of phase U with the flux excited by phase W is calculated.

$$\Psi_{UW} = L_{hUW} \cdot i_W = \cos(4\pi/3) \cdot L_{hUU} \cdot i_W = -\frac{L_{hUU}}{2} \cdot i_W \quad . \tag{4.53}$$

For a symmetrical three-phase system, it is: $i_U + i_V + i_W = 0$. Hence, with $i_U = - i_V - i_W$, the resultant magnetising inductance L_{hU} of phase U is:

$$\Psi_U = L_{hU} i_U \tag{4.54}$$

$$\Psi_U = L_{hUU} i_U - \frac{L_{hUU}}{2} \cdot i_V - \frac{L_{hUU}}{2} \cdot i_W = L_{hUU} i_U - \frac{L_{hUU}}{2} \cdot (i_V + i_W) = \frac{3}{2} \cdot L_{hUU} \cdot i_U \tag{4.55}$$

Hence, the **magnetising inductance of one phase** for $m = 3$ is is given by (4.56), what is in accordance with (4.46):

$$L_{hU} = \frac{3}{2} L_{hUU} = \mu_0 N^2 k_{w,1}^2 \frac{2 \cdot 3}{\pi^2}\frac{l\tau_p}{p \cdot \delta} \tag{4.56}$$

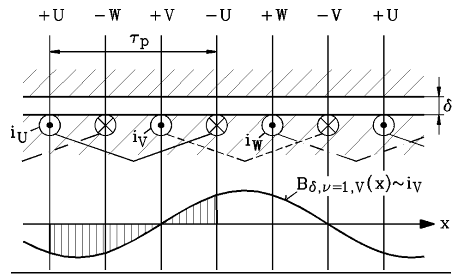


Fig. 4.9: Flux linkage of phase U with the flux excited by phase V

Result:
It was shown that the magnetising inductance per phase of a three-phase system L_h is 1.5 times the self-inductance of an individual phase. This is true for all three phases, because of the symmetry.

Example 4.4-1:
Inductance of a three-phase rotary reactor: The stator has a three-phase winding, the rotor does not have any winding. Per phase, the inductance $L_{h,tot}$ (plus slot and end-winding leakage fields) is effective.

4.5 Mutual Inductance of Two Phases of a Three-Phase Winding

Fig. 4.10a shows a three-phase winding both in the stator and the rotor:
- stator: phases U-X, V-Y, W-Z, index *s*,
- rotor: phases u-x, v-y, w-z, index *r*.
In Fig. 4.10b, the windings are expressed by the inductances L_{ph} that are displaced by $2\tau_p/3$ respectively in the same way as the physical phase windings.

The **rotor** is at stand-still, but it is turned about an angle γ with respect to the stator. The angle γ is between the axes of the windings (middle of the coils) of rotor and stator as shown in Fig. 4.10a. If the rotor is turned about $2\tau_p$ with respect to the stator, the angle γ values 2π. The number of poles of stator and rotor winding **are the same** ($2p$), but the parameter of the phase windings are generally different (Table 4.4).

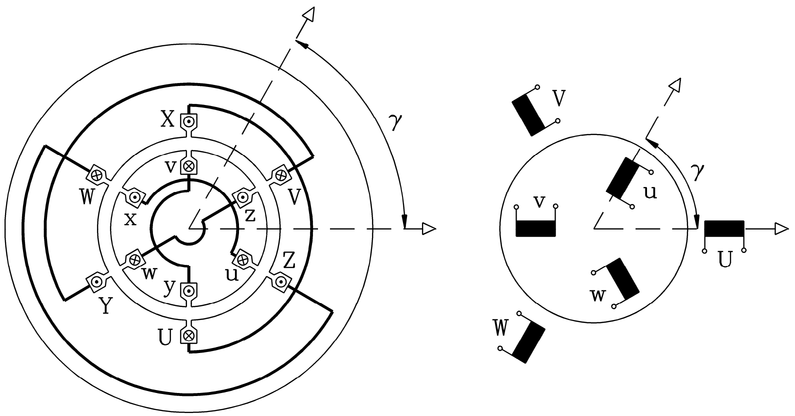
	Stator	Rotor
Number of poles	$2p$	$2p$
Number of phases	m_s	m_r
Number of turns	N_s	N_r
Short-pitching	W_s/τ_p	W_r/τ_p
Number of slots per pole and phase	q_s	q_r
Number of slots	Q_s	Q_r

Table 4.4: Parameters of stator and rotor winding

If the phases of the stator are supplied with a symmetrical three-phase system (stator current I_s, stator frequency f_s), rotating waves travel in the air gap along the circumference, thereby inducing the rotor winding.

$$\boxed{B_{\delta,\nu}(x,t) = \hat{B}_{\delta\nu} \cdot cos\left(\frac{\nu\pi x}{\tau_p} - \omega_s t\right)}, \quad \boxed{\hat{B}_{\delta,\nu} = \frac{\mu_0}{\delta}\frac{\sqrt{2}}{\pi}\frac{m_s}{p}N_s\frac{k_{w,s,\nu}}{\nu}I_s} \qquad (4.57)$$

$$\boxed{\nu = 1 + 2m_s g_s}, \quad \boxed{g_s = 0, \pm 1, \pm 2, \pm 3, ...} \qquad (4.58)$$



a) b)

Fig. 4.10: Three-phase winding of stator and rotor of an electric machine with constant air gap, a) cross-sectional view for a machine with $2p = 2$, $m_s = m_r = 3$, $q_s = q_r = 1$, $W_s = W_r = \tau_p$, b) schematic for arbitrary winding parameters

The amplitudes of the induced voltages (4.59) have to be calculated in analogy to (4.42). The rotor frequency f_r in the rotor **at stand-still** equals the stator frequency: $f_r = f_s$.

$$U_{i,r,\nu} = \sqrt{2}\pi f_s \cdot N_r \cdot k_{w,r,\nu} \cdot \frac{2}{\pi}\frac{\tau_p}{\nu} l\hat{B}_{\delta\nu} \qquad (4.59)$$

The **mutual inductance per phase of the three-phase system $M_{sr,\nu}$** for the field harmonic ν is obtained from (4.48), (4.49):

$$\boxed{U_{i,r,\nu} = \omega_s M_{sr,\nu} I_s} \qquad (4.60)$$

$$\boxed{M_{sr,\nu} = \mu_0 N_s k_{w,s,\nu} N_r k_{w,r,\nu} \frac{2m_s}{\pi^2}\frac{1}{\nu^2 p}\frac{\tau_p l}{\delta}} \qquad (4.61)$$

As the values of $M_{sr,\nu}$ decrease at least with the square of the ordinal number $1/\nu^2$, they quickly become so small that it is sufficient to consider **only the fundamental.**

Example 4.5-1:
Mutual inductance M of a field harmonic: The first relevant harmonic has the ordinal number $\nu = 5$. Therefore, it is $M_{sr,5}/M_{sr,1} < 1/25 = 0.04$.

Because of the rotor shift by γ with respect to the stator, the induced rotor voltages have the voltage **phase shift** γ with respect to the induced stator voltages. This phase shift equals the travelling distance of the rotating waves between the axes of the stator and rotor windings. Further, as the rotor phases u, v, w are also spatially displaced by $2\tau_p/3$, their induced phase voltages are phase shifted by $2\pi/3$. Hence, they also form a symmetrical three-phase system.

Example 4.5-2:
Rotary transformer:
If the phases U and u of the stator and the rotor are connected in series (the same V and v, and W and w), the stator and rotor phase voltages add to each other. The following voltage is induced between the two terminals (input of the stator phase, output of the rotor phase):

$$\underline{U} = \underline{U}_s + \underline{U}_r = U_s + U_r e^{-j\gamma} \qquad (4.62)$$

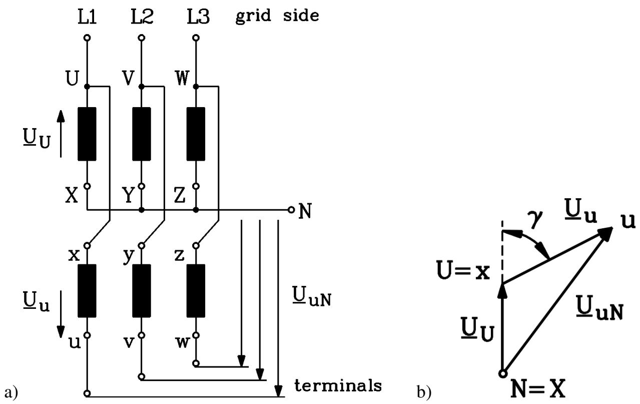
The amplitude and the phase of the resultant voltage can be adjusted continuously by continuous variation of the angle γ. If e.g. the windings of stator and rotor are identically designed, it is $U_r = U_s$ and therefore:

$$\underline{U} = U_s + U_s e^{-j\gamma} = U_s \cdot \left(1 + e^{-j\gamma}\right) \qquad (4.63)$$

At $\gamma = 0$, the value of the stator voltage is doubled to $2U_s$, at $\gamma = \pi$ it is zero.

Result:
The rotary transformer allows continuous variation of the voltage from 0 up to $2U_s$.

Rotary transformers ("induction regulators") are often used in test floors, where a continuously adjustable amplitude of the voltage is used e.g. for the measurement of the characteristic curves at no-load and at short-circuit of induction machines or transformers.



a) b)

Fig. 4.11: Rotary transformer: a) principle set-up, b) voltage generation